NEW YORK UNIVERSITY
WASHINGTON SQUARE COLLEGE OF ARTS AND SCIENCE
MATHEMATICS RESEARCH GROUP

RESEARCH REPORT No. TW-14

THE GAIN AND THE NOISE FIGURE OF THE HELICAL TRAVELING WAVE TUBE

by

WILLIAM SOLLFREY

and

MORTIMER WEITZ

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William Sollfrey
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Written by:

William Sollfrey
William Sollfrey

Mortimer Weitz
Mortimer Weitz

Morris Kline
Morris Kline
Project Director

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Abstract

Previous investigators have obtained expressions for the gain and noise figure of a helical traveling-wave tube. We consider in detail the dependence of these quantities upon the various tube parameters, that is, the beam current, beam voltage, beam radius, and frequency. The restrictive assumptions are made that the tube is tuned to resonance and that the ranges of the parameters are so limited that all the Bessel functions which appear have large arguments. These assumptions are lifted in the last two sections of the paper. The general course of the curve of gain or noise figure versus the parameter being varied is studied analytically and the results presented graphically.

1. Introduction

The gain and the noise figure of a helical traveling wave tube were first investigated by Pierce[1], under severely restrictive assumptions. His theory was then improved by Chu and Jackson[2], who use a quite accurate field theory to obtain a transcendental equation, the roots of which are the propagation constants of the various modes. In order to obtain approximate solutions of this equation, they fix all the parameters of the problem. Hence their work gives no indication of how the propagation constants very as the parameters are changed.

Further research on this equation has been carried out by Friedman[3] who studied the gain problem. He assumed that the modes of the helix in the presence of the beam differ only by small perturbations from the modes of the "cold" helix. An expansion procedure may then be applied to obtain the propagation constants. If the ranges of the system parameters are restricted so as to satisfy the resonance condition and also so as to yield large arguments in all Bessel functions that appear, then the dependence of the propagation constants on the parameters may be easily investigated.

In Pierce's early paper, an estimate for the noise figure is given, based primarily on circuit analysis. We shall apply Friedman's improved analytical methods to this estimate, and then, under the same restrictions which apply to the gain, obtain the dependence of the noise figure on the system parameters. By this means we remove Pierce's assumption that the electron beam is treated as an infinitely thin shell, without using the "equivalent thin beam" treatment of Fletcher[4]. The noise figure has been treated by Shulman and Heagy[5], using analytical methods similar to Friedman's but retaining the assumption of an infinitely thin beam.

This paper may thus be considered as extensions and applications of Friedman's work. By use of the improved methods, we obtain analytical expressions for the gain and noise figure, from which we may treat the explicit dependence on the parameters. The details of this dependence are investigated

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1. J. R. Pierce, "Theory of the Beam Traveling Wave Tube", P.I.R.E. 35, 111 (1947).
2. L. J. Chu & J. D. Jackson, "Field Theory of Traveling Wave Tubes", P.I.R.E. 36, 853 (1948)
3. B. Friedman, "Amplification of the Traveling Wave Tube", N.Y.U. Math Research Group Report No. TW-9.
4. R. C. Fletcher, "Traveling Wave Tube Helix Parameters", P.I.R.E. 38, 413 (1950)
5. Carl Shulman & M. S. Heagy, "Small-Signal Analysis of the Traveling Wave Tube", R.C.A. Review 8, 585 (1947.

by varying each parameter separately. The general course of the curves of gain and noise figure against the parameter being varied is treated analytically, and the results presented graphically. We compare the theory with experiment when possible.

In the next section we shall summarize the pertinent results of Chu and Jackson and of Friedman. In Section 3 an expression for the noise figure is obtained by substituting the Chu-Jackson-Friedman propagation constants for those of Pierce in the latter's estimate of the noise figure. In Section 4 the behavior of the gain and noise figure is studied when the beam current is varied while the other parameters are maintained constant. The voltage alone is varied in Section 5, while the beam thickness is the variable parameter in Sections 6 and 7. In the eighth section the frequency is varied, which yeilds the frequency response and effective band width of the tube, as far as the latter is determined by the helical circuit rather than the external circuit. The effect of small departures from resonance is investigated in the next section, and in Section 10 we discuss lifting the requirement of large arguments in the Bessel functions.

Although the validity of the gain and noise formulae studied herein is asserted only under restrictive assumptions, it is not unreasonable to expect that wider application would yield results which are at least qualitatively correct.

2. Summary of the Pertinent Results of Chu and Jackson and of Friedman

To lay the groundwork for our analysis, we shall first summarize those results of Chu and Jackson and of Friedman which we shall require. Since the system composed of the helix plus the beam is far too complicated to solve directly, a set of physical assumptions are made to simplify the problem. First, one assumes that the helix is well approximated by a lossless helical sheath of radius a and infinitesimal thickness, which sheath is capable of conduction only in a direction which makes an angle of $90^\circ - \theta$ with the axis of the helix. Second, one assumes that because of the presence of a strong longitudinal magnetic field, the electrons are constrained to have only longitudinal motion, and to be confined within a cylinder of radius b concentric with the axis of the helix. Third, the d.c. beam current density is assumed to be constant over the beam cross-section, and since there is no radial motion, this constant density is maintained over the length of the beam. Lastly, all a.c. quantities are assumed to be small compared to the corresponding d.c. quantities, which assumption linearizes the equations of motion and enables a solution to be obtained. This last assumption implies that we do not deal with such great lengths of beam that the amplified currents become comparable in magnitude to the d.c. beam current.

With circular symmetry around the z-axis, taken to lie along the axis of the helix, solutions of the field equations can be found for the three regions $r < b$, $b < r < a$, and $r > a$. The boundary conditions at the beam require the radial admittance function to be continuous, and the boundary conditions at the helix then enable the other constants involved to be determined. If one assumes the fields vary as $e^{i\omega t - \gamma z}$, the propagation constant γ satisfies the transcendental equation:

$$(2.1) \qquad i\,\frac{ka}{pa}\,\frac{\lambda}{p}\,\frac{I_1(\lambda b)}{I_0(\lambda b)} = i\,\frac{ka}{pa}\left[\frac{I_1(pb) - G(pa)K_1(pb)}{I_0(pb) + G(pa)K_0(pb)}\right]$$

in which we have used the following notation.

(2.2)

$k^2 = \omega^2\mu\epsilon \qquad k = 2\pi/\lambda = \omega/c$

$p^2 = -(\gamma^2 + k^2)$

e - electron charge

m - electron mass

v_0 - d.c. beam velocity

I - d.c. beam current

$$(2.3) \qquad \eta^2 = p^2 \left[1 + \frac{\frac{e}{m} I}{\pi b^2 \epsilon v_0^3 (1\frac{\omega}{v_0} - \mu)^2} \right] \quad .$$

I_0, I_1, K_0, and K_1 are modified Bessel functions, and $G(pa)$ is defined as :

$$(2.4) \qquad G(pa) = \left[\left(\frac{ka \cot \theta}{pa}\right)^2 I_1(pa)\, K_1(pa) - I_0(pa)\, K_0(pa) \right] / K_0^2(pa) \quad .$$

At this point Chu and Jackson fix all the parameters of the system, and then solve the equation numerically. Friedman's procedure is to assume that the modes of the system including the beam will differ only slightly from the modes of the system with the beam absent. He therefore writes:

$$pa = p_0 a + y = x_0 + y$$

where $x_0 = p_0 a$ is the real positive root of $G(pa) = 0$ and thus is the solution in the absence of the beam. The two sides of equation (2.1) are expanded in powers of y. Since we are interested in the case when the beam velocity and the phase velocity of the propagating mode of the cold helix are nearly equal, the denominator $(1\frac{\omega}{v_0} - \mu)^2$ in η must be treated more accurately in the expansion than the other places where y appears. There results upon keeping lowest order items:

$$(2.5) \qquad y(y - d_0)^2 - \zeta\, (1 + \delta y) = 0$$

in which we have introduced the further symbols:

$$d_0 = \frac{\omega a}{v_0} - x_0$$

(2.6)

$$W = I/2\pi\epsilon \sqrt{2e/m}\; V^{3/2} = 3.03 \times 10^4\, I/V^{3/2} \qquad V - \text{d.c. beam voltage}$$

$$\delta = \frac{bx_0}{a} \left[\frac{I_0\left(\frac{bx_0}{a}\right)}{I_1\left(\frac{bx_0}{a}\right)} - \frac{I_1\left(\frac{bx_0}{a}\right)}{I_0\left(\frac{bx_0}{a}\right)} - \frac{1}{\frac{bx_0}{a}} \right]$$

$$\zeta = x_0 \, I_0\left(\frac{bx_0}{a}\right) I_1\left(\frac{bx_0}{a}\right) (1 + \delta) \, W/2 \, \frac{b}{a} \, G'(x_0)$$

(2.6)

$$\xi = \frac{K_0\left(\frac{bx_0}{a}\right) G'(x_0)}{I_0\left(\frac{bx_0}{a}\right)}$$

The quantities here are interpreted straightforwardly. The departure from resonance between the beam velocity and the helical mode phase velocity is represented by d_0. Throughout most of this work we shall set $d_0 = 0$. The amplification has a broad maximum near $d_0 = 0$, so this approximation should not affect matters much. W, which is proportional to the beam current, is a measure of the strength of space-charge forces. The quantities δ and ξ, which depend primarily on $\frac{bx_0}{a}$, are geometrical factors, while ζ measures the combined effect of geometry and the presence of the beam.

Since we are dealing with the modes of the system which reduce to the propagating mode of the cold helix when I approaches zero, we shall assume that the phase velocity of that mode is much less than the velocity of light, whence we may obtain the propagation constants of the modes by the approximation $\gamma = ip$. We therefore can get the propagation constants from the solutions of equation (2.5). When the tube is amplifying, two of the roots of (2.5) will be complex conjugates. We shall call y_1 the root which represents the amplifying mode, and y_2 the root corresponding to the attenuated mode. The third root y_3, which is real, corresponds to a mode which is neither amplified nor attenuated. The amount of amplification is proportional to the imaginary part of y_1. Our considerations about the gain will therefore revolve about this quantity.

Friedman further considers the case that $d_0 = 0$ and that the Bessel functions have sufficiently large arguments, say $bx_0/a>3$, that the use of the asymptotic formulas for the Bessel functions is justified. The satisfying of these conditions will insure satisfying $\gamma = ip$. The constants in equation (2.5) can be simplified under these restrictions, yielding

(2.7) $$y^3 - Ly + \frac{L}{A} = 0$$

where

$$A = \frac{2}{x_0} e^{2x_0(1-\frac{b}{a})}$$

(2.8)

$$L = \frac{1}{4}\left(\frac{a}{b}\right)^2 W = .77 \times 10^4 \left(\frac{a}{b}\right)^2 \frac{I}{V^{3/2}} \quad .$$

Upon setting $y = (L/A)^{1/3} u$, $B = (A^2L)^{1/3}$, (2.7) becomes:

(2.9) $$u^3 - Bu + 1 = 0 \quad .$$

The condition that (2.9) have complex roots is $B < 3/\sqrt[3]{4} = 1.89$. For larger values of B no amplification will take place. Since B is proportional to $I^{1/3}$, this condition indicates the existence of a limiting current above which no amplification takes place.

For small B Friedman gives approximate solutions of (2.9). To the lowest order in B these solutions are:

$$u_1 = \left(\frac{L}{A}\right)^{-1/3} y_1 = \frac{1}{2}\left[1 + \frac{B}{3} + i\sqrt{3-2B}\right]$$

(2.10) $$u_2 = \left(\frac{L}{A}\right)^{-1/3} y_2 = \frac{1}{2}\left[1 + \frac{B}{3} - i\sqrt{3-2B}\right]$$

$$u_3 = \left(\frac{L}{A}\right)^{-1/3} y_3 = -1 - \frac{B}{3} \quad .$$

The propagation constants of the three modes are thus given to the present degree of approximation by:

$$\gamma_1 = \frac{1}{a}\left[x_0 + \frac{1}{2}\left(\frac{L}{A}\right)^{1/3}\left\{1 + \frac{B}{3} + i\sqrt{3-2B}\right\}\right]$$

(2.11) $$\gamma_2 = \frac{1}{a}\left[x_0 + \frac{1}{2}\left(\frac{L}{A}\right)^{1/3}\left\{1 + \frac{B}{3} - i\sqrt{3-2B}\right\}\right]$$

$$\gamma_3 = \frac{1}{a}\left[x_0 - \left(\frac{L}{A}\right)^{1/3}\left(1 + \frac{B}{3}\right)\right] \quad .$$

The gain per unit length of the tube is accordingly given by the real part of γ_1, or

(2.12)
$$G = \frac{1}{2a}\left(\frac{L}{A}\right)^{1/3} \sqrt{3-2B} \quad .$$

In terms of the system parameters, this may be written as:

(2.13)
$$G = \frac{1}{4}\, c_1 \frac{I^{1/3}\, x_o^{1/3}\, e^{-2/3\, x_o(1-\frac{b}{a})}}{V^{1/2}\, a^{1/3}\, b^{2/3}}\left[3 - \frac{2c_1 I^{1/3} a^{2/3} e^{4/3\, x_o(1-\frac{b}{a})}}{V^{1/2}\, b^{2/3}\, x_o^{2/3}}\right]^{\frac{1}{2}}$$

in which we have introduced the constant:

(2.14)
$$c_1 = \left[2\pi\epsilon\sqrt{\frac{2e}{m}}\right]^{-1/3} = 31.2 \quad .$$

The dependence of G on the various parameters will be studied in later sections. We shall next obtain a similar expression for the noise figure.

3. The Noise Figure Formula.

The noise figure of an amplifier is conventionally defined as the ratio of the actual output noise power to the output noise power that would be present if the amplifier were ideal, that is, if it merely amplified the thermal noise at the input, where one assumes the input is matched to a source at temperature T. Pierce assumes that the only source of excess noise in the traveling-wave tube is shot noise in the beam, neglecting the partition noise arising from electrons striking the helix. This latter effect will be small if b/a is not too near unity and if the magnetic field is sufficiently strong. If the tube is in the amplifying range and if the helix is sufficiently long, then essentially all the power at the output is in the amplified mode. Thus, the noise figure is found by comparing the power in this mode produced by the shot noise, treated as an input current, to the power produced by the thermal noise, treated as an input electric field. Pierce finds after some computation:

$$F = \frac{E_n^2}{E_t^2} = \frac{8\pi\Gamma^2 eV^2 v_o^2}{\eta k_1 T\omega^2 I} \left|\left(\gamma_2 - \frac{i\omega}{v_o}\right)\left(\gamma_3 - \frac{i\omega}{v_o}\right)\right|^2 \tag{3.1}$$

$$\frac{\sqrt{k^2+\left(\frac{x_o}{a}\right)^2}}{x_o^2}\, ka^4 \left[\left(1+\frac{I_oK_1}{I_1K_o}\right)\left(I_1^2-I_oI_2\right)+\left(\frac{I_o}{K_o}\right)^2\left(1+\frac{I_1K_o}{I_oK_1}\right)\left(K_oK_2-K_1^2\right)\right] .$$

Here the Bessel functions are all to be evaluated at x_o. We have introduced the further notation $\eta = 120\pi$ ohms, the intrinsic impedance of free space, and k_1 is Boltzmann's constant, which is usually denoted by k, but we have reserved k for another purpose. By Γ^2 we refer to the space charge reduction factor whereby the shot noise in a beam is reduced below the value it would have if space charge forces were neglected. Specifically, the mean square noise current in the beam in the frequency band B is given by:

$$i^2 = 2e\Gamma^2 IB \quad . \tag{3.2}$$

We now wish to extend Pierce's analysis by using Friedman's formulas for the propagation constants. We assume the system is tuned to resonance, so $d_o = 0$, and that bx_o/a is large, so we may use the asymptotic formulas for the Bessel functions. The Bessel function factor in (3.1) may be simplified under the weaker assumption x_o is large, which is always satisfied in practice. We then have:

$$(3.3)\qquad F = \frac{4\Gamma^2 m^2 v_o^6}{\zeta k_1 T e \omega^2 I}\ \frac{\left|y_2 y_3\right|^2 k e^{2x_o}}{a\, x_o^3}\quad .$$

Upon using equations (2.10) for the propagation constants, this becomes:

$$(3.4)\qquad F = \frac{4\Gamma^2 m^2 v_o^6}{\zeta k_1 T e \omega^2 I}\ \frac{k e^{2x_o}}{a x_o^3}\left(\frac{L}{A}\right)^{4/3}\left(1 + \frac{B}{3}\right)\quad .$$

We use $\frac{1}{2} m\, v_o^2 = e\, V$ to eliminate the beam velocity, and the definitions of L, A, and B to express F directly in terms of the system parameters. Because of the assumption the tube is tuned to resonance, the parameters are restricted by the condition $V = \frac{1}{2}\ \frac{mv_o^2}{e} = \frac{1}{2}\ \frac{m}{e}\left(\frac{\omega a}{x_o}\right)^2$, which is equivalent to $v_o = c \sin\theta$, the well-known resonance condition. We also assume that the temperature of the source is fixed and that the electron gun is so designed that Γ^2, which depends only upon gun design, is independent of the other parameters, and hence may be treated as a constant. We then have for F:

$$(3.5)\qquad F = \frac{c_2 V I^{1/3} a^{5/3} e^{2/3}\, x_o(4\frac{b}{a} - 1)}{\omega\, x_o^{5/3}\, b^{8/3}}\left[1 + \frac{1}{3}\ \frac{c_1 I^{1/3} a^{2/3} e^{4/3}\, x_o(1 - \frac{b}{a})}{V^{1/2}\, b^{2/3}\, x_o^{2/3}}\right]$$

in which we have introduced the additional symbol c_2 which is independent of all the system parameters and hence may be treated as a constant:

$$(3.6)\qquad c_2 = \frac{1}{2\pi^{4/3}}\cdot\frac{\Gamma^2}{k_1 T}\ \frac{e^{4/3}}{(m\epsilon)^{1/3}} = 1.15 \times 10^8\,\Gamma^2$$

in which we have used the standard reference temperature R = 300°K. In general Γ^2 will not differ too greatly from unity. The manner in which F depends on the various indicated parameters will make up a large portion of the remainder of this work.

Because F and G depend on the system parameters in a very complicated manner, we shall consider the variation of F and G on each parameter separately. To save much writing we shall use the notation that the letter c with a subscript refers to a quantity which is constant through the section in which it is introduced, as distinct from c_1 and c_2, which are fixed constants throughout the work. In general, the c quantities from c_3 on will be complicated functions of those parameters which are not being varied.

We shall first investigate how F and G depend on current.

4. Dependence of Gain and Noise Figure on Current

Suppose that in (2.13) and (3.5) we keep a, b, ω, and V fixed, and consider that only I is variable. The derived parameter $x_o = \omega a/v_o$ will also be constant. F and G then reduce to:

$$\text{(4.1)} \quad \begin{aligned} &\text{a)} \quad G = c_3 I^{1/3}\left[\frac{3}{2} - c_4 I^{1/3}\right]^{1/2} \\ &\text{b)} \quad F = c_5 I^{1/3}\left[1 + c_6 I^{1/3}\right] \quad . \end{aligned}$$

$$c_3 = c_1 x_o^{1/3} e^{-2/3\, x_o(1-\frac{b}{a})} / 2^{3/2} V^{1/2} a^{1/3} b^{2/3}$$

$$c_4 = c_1 a^{2/3} e^{4/3\, x_o(1-\frac{b}{a})} / V^{1/2} b^{2/3} x_o^{2/3}$$

$$c_5 = c_2 V a^{5/3} e^{2/3\, x_o(4\frac{b}{a}-1)} / \omega x_o^{5/3} b^{2/3}$$

$$c_6 = \frac{1}{3} c_4 \quad .$$

We first discuss the dependence of G on I. At I = 0, G is zero, while its first derivative is infinite. As I increases, G increases, attaining a maximum at $I = 1/c^{3/4}$, that is, at $B = c_4 I^{1/3} = 1$. The value of the gain at the maximum is:

$$\text{(4.3)} \qquad G_{IM} = \frac{1}{4}\,\frac{x_o}{a}\, e^{-2x_o(1-\frac{b}{a})} = \frac{1}{a}\cdot\frac{.50}{A}$$

using the double subscript IM to denote the maximum when I alone is varied. A more accurate calculation, to be performed in Section10, changes the value of the coefficient from .50 to .58. The same calculation shows that the maximum actually occurs at B = 1.24 instead of B = 1.

As I increases further, the gain decreases, and it passes through zero with infinite slope at $I = (3/2c_4)^3$, or B = 3/2. The more accurate calculation shifts this value to B = 1.89. Hence, when the current increases beyond a critical value, the gain begins to fall off. This may be expected from the following elementary reasoning. The gain arises from the strong bunching action of the fields on the beam, caused by the electrons remaining in phase with the traveling waves. When the d.c. current becomes large, strong space charge forces act on the electrons, and these are in such a direction as to oppose the bunching

process, thereby causing a decrease in the rate of change of gain with current. For sufficiently high currents, the space charge forces overpower the field forces, and the gain decreases to zero as the current is further increased. This behavior was predicted independently by Friedman[3] and Benedict[6].

As far as the noise figure is concerned, when a, b, ω, and V are kept fixed, it takes the form (4.1b). This function starts from zero at I = 0 with infinite slope, and then increases monotonically as I increases. Hence the noise figure gets progressively worse with increasing surrent. A quantity which may designate a figure of merit for the tube is the ratio of gain to noise figure. This quantity R takes the form:

$$R = \frac{G}{F} = \frac{c_3}{c_5} \frac{\left[\frac{3}{2} - c_4 I^{1/3}\right]^{1/2}}{\left[1 + \frac{1}{3} c_4 I^{1/3}\right]} \qquad (4.4)$$

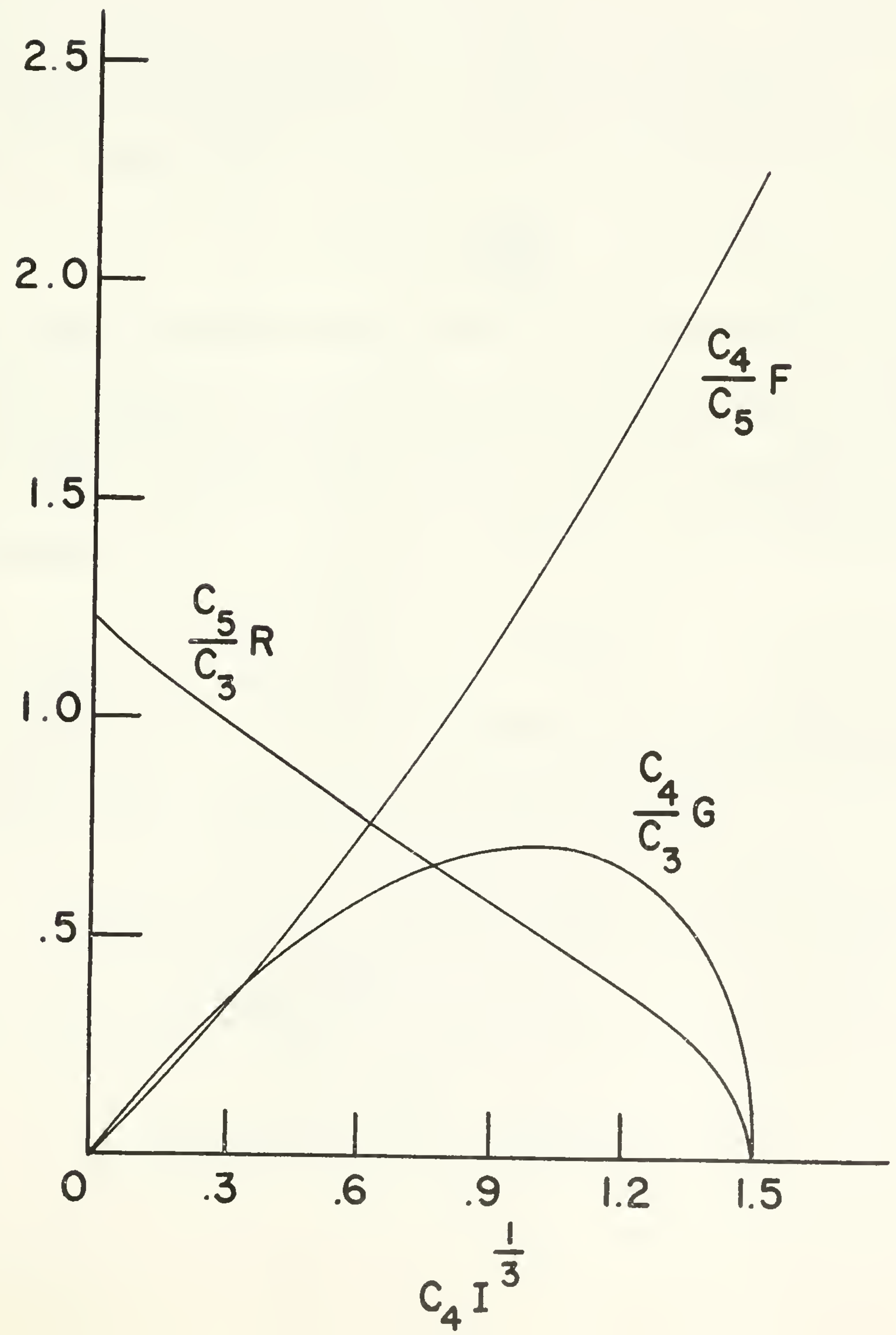
As the current increases, R decreases monotonically from the value $\sqrt{\frac{3}{2}}\,\frac{c_3}{c_5}$ at I = 0 to zero at $I = \left(\frac{3}{2c_4}\right)^3$. Hence, the effectiveness of the tube in producing amplification of signal with respect to noise decreases steadily as the current increases.

Graphically, G, F, and R are depicted in Figure 1.

6. D. L. Benedict, "Ballistic Treatment of the Traveling-Wave Amplifier", Harvard Univ. Cruft Lab. Report No. 30.

Figure 1

Gain, Noise Figure, and Merit Figure versus Current



5. Dependence of Gain and Noise Figure on Beam Voltage.

The dependence of the gain and noise figure on beam voltage is very complicated. This is because the tuning to resonance of the tube depends on beam velocity, which has to be matched to the phase velocity of the propagation mode of the cold helix. Hence, variations in beam velocity will produce departures from resonance if ω, a, and x_o are kept fixed. However, the basic cubic equation (2.5) has a broad maximum as d_o varies from zero. Hence, we shall continue to use equations (2.13) and (3.5), which hold strictly only at resonance, to describe the behavior away from resonance. The specific effect of the departures from resonance will be studied in Section 9.

To describe the dependence on voltage, we shall introduce the auxiliary variable z, defined by:

$$z = \frac{bx_o}{a} = \frac{b\,\omega}{\sqrt{2\frac{e}{m}V}} \tag{5.1}$$

From the assumptions under which (2.13) and (3.5) were derived, z must be greater than 3. All our considerations will accordingly be restricted to this "domain of validity". We note that increasing z corresponds to decreasing V, and the domain of validity extends from z = 3 to z = ∞, or from V = 0 to $V = b^2\omega^2 m/18e$. This latter value corresponds to a beam velocity of the order of magnitude:

$$\frac{v_o}{c} \sim \frac{2b}{\lambda} \tag{5.2}$$

which is small for all practical tubes.

Equation (2.13) for the gain, in addition to direct dependence upon z, involves V explicitly. V may be eliminated by using (5.1), thereby expressing the gain in terms of z and "constants". There results:

$$G = c_7\, z^{4/3}\, e^{-2/3\, z(\frac{a}{b} - 1)} \left[1 - c_8\, z^{1/3}\, e^{4/3\, z(\frac{a}{b} - 1)}\right]^{1/2}$$

$$c_7 = \frac{\sqrt{6}}{4}\, \frac{c_1 I^{1/3}\sqrt{e/m}}{b^2} \tag{5.3}$$

$$c_8 = \frac{2\sqrt{2}}{3}\, \frac{c_1\, I^{1/3}\sqrt{e/m}}{b} = \frac{8}{3\sqrt{3}}\, bc_7$$

The constants c_7 and c_8, which are both proportional to $I^{1/3}$, will be quite small for low currents. Also, c_8 may be made small with respect to c_7 by using small values of b.

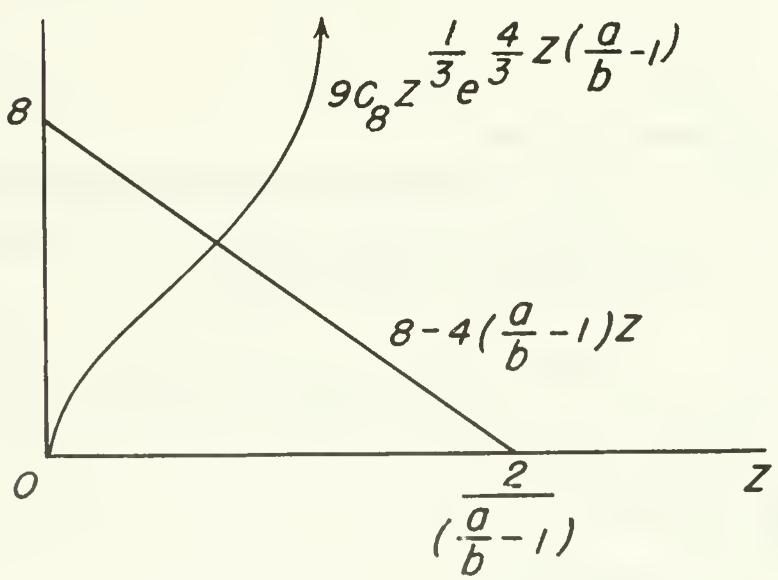
The range of values of z for which G is positive is from z = 0 to some value z_m at which the radical vanishes. To determine the number of maxima or minima within the range we set the derivative of G with respect to z equal to zero, and obtain the transcendental equation:

$$(5.4) \qquad 9c_8\, z^{1/3}\, e^{4/3\, z(\frac{a}{b} - 1)} = 8 - 4\,(\frac{a}{b} - 1)\, z \quad .$$

The first of these functions starts up from zero at z = 0 with infinite slope, and thereafter is monotone increasing. The second function slopes downward from the value 8 at z = 0 to zero at $z = 2/(\frac{a}{b} - 1)$. Hence, as is clear from the accompanying diagram, Figure 2, equation (5.4) has only one root.

Figure 2

Graphical Solution of Equation (5.4)



Consequently, G has only one maximum in the range. It thus starts at zero, rises to a maximum value G_{zm}, and then decreases to zero at $z = z_m$.

However, for these considerations to be valid, z must lie in the domain of validity $z > 3$. We discuss the question in two steps. First, is z_m less than or greater than 3? If z_m is less than 3, the amplifying region described by $0 < z < z_m$ is entirely outside the domain of validity, and the considerations are meaningless. There exists a critical value, $c_g = c_{gk}$, for which the radical in (5.3) vanishes for $z = 3$. This value is given by:

$$c_{gk} = 3^{-1/3} e^{-4(\frac{a}{b} - 1)} \quad . \tag{5.5}$$

For larger values of c_g, the amplifying range is entirely outside the domain of validity. Hence, we must restrict ourselves to c_g less than c_{gk}. When c_g is less than c_{gk}, the portion of the curve of G against z from $z = 3$ to $z = z_m$ lies in the domain. Now the curve of G against z rises from 0 at $z = z_m$ with infinite slope, and as z decreases, G increases until the maximum value G_{zm} is reached. Thus, the next question is, does the maximum value lie within the domain of validity?

Figure 2 indicates that the maximum must occur for a value of z less than $2/(\frac{a}{b} - 1)$. If this latter quantity is less than 3, the maximum must be outside the domain of validity. Now $2/(\frac{a}{b} - 1)$ is equal to 3 when b/a equals 3/5. Hence for b/a less than 3/5, the maximum is outside the domain of validity, and the included portion of the curve of G against z is monotonic.

In case b/a is greater than 3/5, the position of the maximum still depends upon c_g. If c_g is sufficiently large, the intersection point on Figure 2 may be brought close to zero, while for small values of c_g it may be brought close to $2/(\frac{a}{b} - 1)$. As c_g decreases, the intersection point shifts steadily to the right, and if $2/(\frac{a}{b} - 1)$ is greater than 3, which is satisfied for $b/a > 3/5$, there will be a critical value $c_g = c_{gc}$ for which the intersection point occurs at $z = 3$. Consequently, for $c_g > c_{gc}$, the maximum lies outside the domain at validity, and the included portion of the curve is monotonic. For $c_g < c_{gc}$, the curve rises to a maximum and then decreases. Since c_{gc} is defined in such a way that for $c_g = c_{gc}$ Equation (5.4) is valid for $z = 3$, c_{gc} is given by:

$$c_{gc} = \frac{\frac{8}{3} - 4\,(\frac{a}{b} - 1)}{3^{4/3}\, e^{4(\frac{a}{b} - 1)}} \quad . \tag{5.6}$$

We now reverse these considerations to describe the variation of G with the beam voltage V. Recapitulating shows the following:

For $c_8 > c_{8k}$, the amplification range is outside the domain of validity. For $c_8 < c_{8k}$ and $b/a < 3/5$, the gain starts from zero at $V = V_m$ ($V = V_m$ corresponds to $z = z_m$), and increases monotonically as V increases until we reach the end of the domain of validity. This behavior continues to hold for $b/a > 3/5$, but $c_{8c} < c_8 < c_{8k}$. Finally, for $b/a > 3/5$ and $c_8 < c_{8c}$, the gain rises from zero at $V = V_m$ to a maximum value and then decreases as V increases further. This behavior is sketched in Figure 3.

Now from its definition, c_8 has the value:

$$(5.7) \qquad c_8 = \frac{2\sqrt{2}}{3}\, c_1 \sqrt{e/m}\, \frac{I^{1/3}}{\omega b} = 1.24 \times 10^7 \, \frac{I^{1/3}}{\omega b} \quad .$$

Consequently, c_8 is effectively proportional to $I^{1/3}$, since b and ω are not often varied. Therefore, the analysis shows that if the current is sufficiently small and $b/a > 3/5$, the amplification rises from zero at $V = V_m$ to a maximum amplification and then decreases. When the current is increased to such a value that $c_8 > c_{8c}$, the only portion of the G - V curve about which we can speak, that included in $z > 3$, is a monotone increasing function of V.

We shall next consider the noise figure. Equation (3.5) for the noise figure may be transformed to depend only upon z by eliminating the explicit V-dependence. It becomes:

$$F = c_9\, e^{2/3\, z(4 - \frac{a}{b})} z^{-11/3} \left[1 + c_{10}\, z^{1/3}\, e^{4/3\, z(\frac{a}{b} - 1)} \right]$$

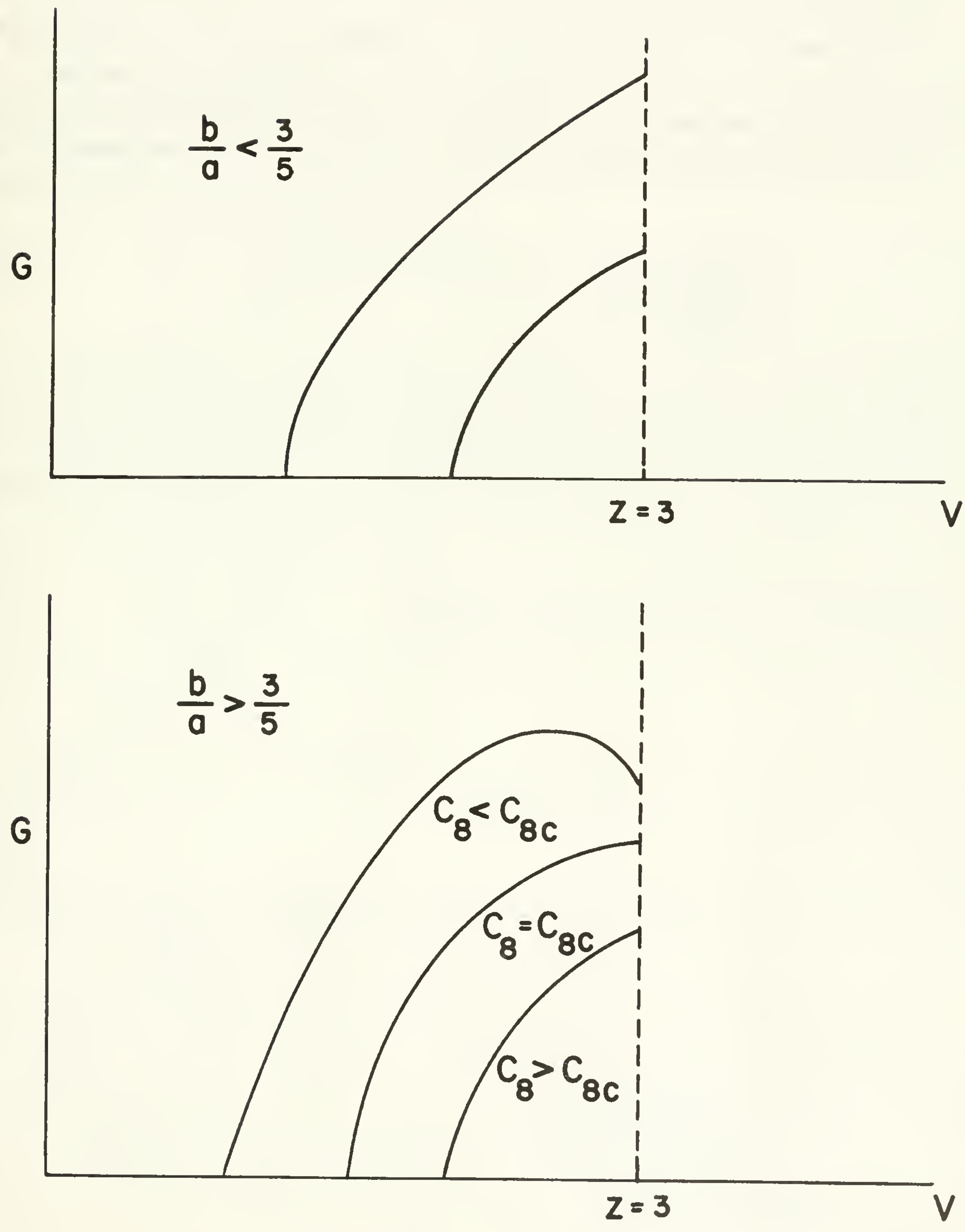
$$(5.8) \qquad c_9 = c_2\, \frac{m}{2e}\, I^{1/3}\, \omega b$$

$$c_{10} = \frac{\sqrt{2}}{3}\, c_1 \sqrt{e/m}\, I^{1/3}/\omega b = \frac{1}{2}\, c_8 \quad .$$

We are interested in the variation of F with z within the amplifying range. To this point we have worked with the approximate equation (2.12) for the gain, which asserts that the limiting value of B is 1.5. Now the more accurate theory shows the limiting value of B is actually 1.89. Hence, the second term in the bracket in (5.8), which equals B/3, lies between zero and .63 as z varies from zero to z_m. The domain of validity extends from $z = 3$ to $z = z_m$. We accordingly may define a critical value $c_{10} = c_{10k}$ for which $z_m = 3$, and restrict outselves to values of c_{10} less than c_{10k}. The critical value is given by:

Figure 3

Sketch of Gain versus Beam Voltage for Various Beam Currents



$$(5.9) \qquad c_{10k} = .63 . 3^{-1/3}\, e^{-4(\frac{a}{b} - 1)} = .437\, e^{-4(\frac{a}{b} - 1)} \quad .$$

We should have $c_{10k} = \frac{1}{2}\, c_{8k}$, but the use of the more accurate theory changes the factor to .63. Actually c_{10k} as given here is correct, but c_{8k} as defined by (5.5) should have a factor 1.26. The use of the correct limiting value 1.89 instead of 1.5 may be expected to improve formulas (2.13) and (3.5).

Regarded as a function of z, F is infinite at $z = 0$, is positive for $z > 0$, and is again infinite for $z = \infty$. To investigate the number of maxima and minima, we differentiate F with respect to z, and set the derivative equal to zero, which yields the transcendental equation:

$$(5.10) \qquad c_{10}\, z^{1/3}\, e^{4/3\, z(\frac{a}{b} - 1)} = - \frac{(8 - 2\frac{a}{b})z - 11}{(4 + 2\frac{a}{b})z - 10} \quad .$$

The left hand member of this equation was sketched in Figure 2. The behavior of the right hand side depends upon whether b/a is less than or greater than 1/4, since the coefficient of z in the numerator changes sign at b/a = 1/4. When b/a is less than 1/4, we have the behavior illustrated in Figure 4.

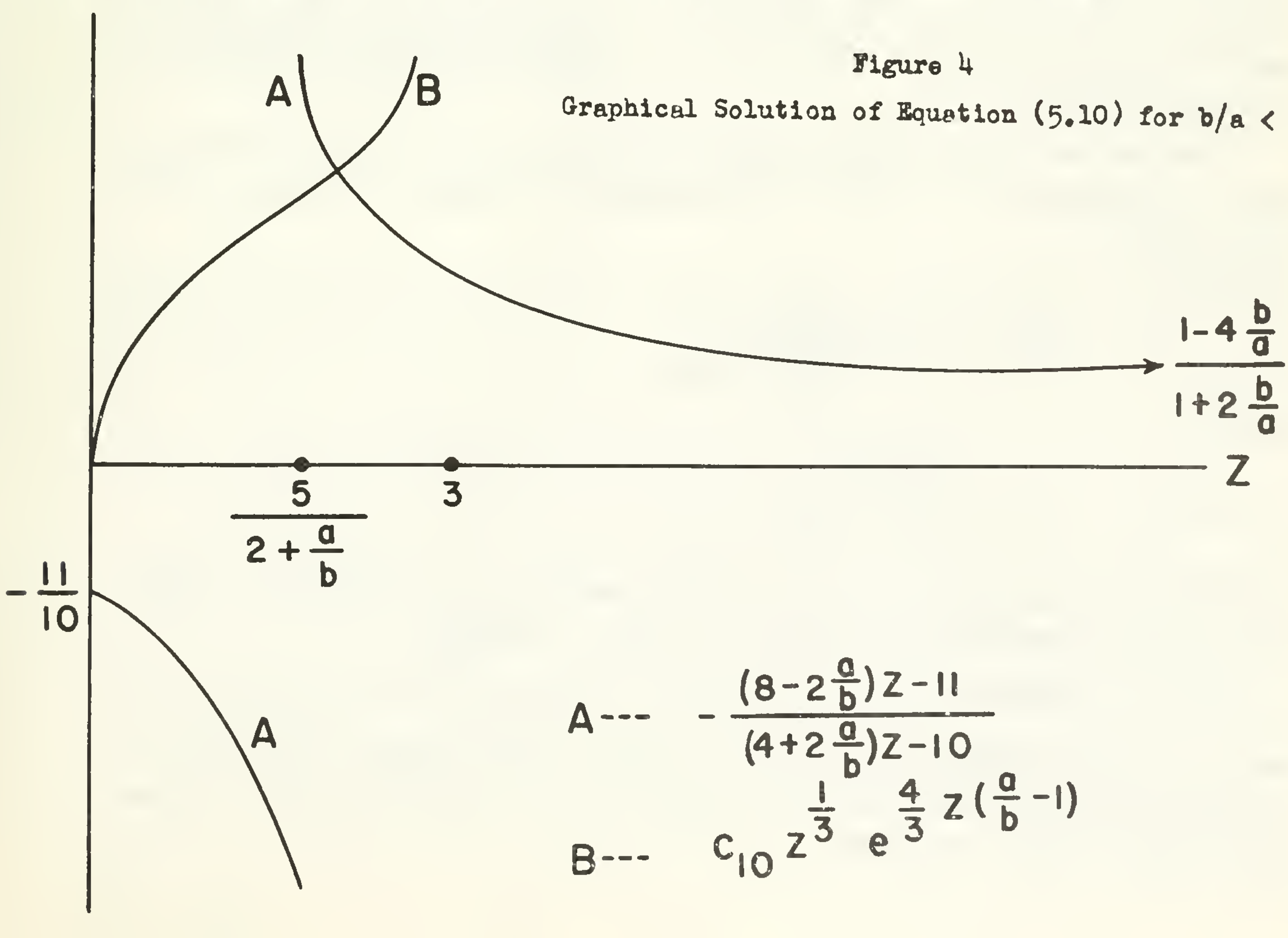


Figure 4

Graphical Solution of Equation (5.10) for b/a < 1/4

Consequently, there is only one root of (5.10) for z positive. Therefore F has only one stationary point, and this stationary point must be a minimum. The slope of the F - z curve is negative for z less than the z corresponding to the intersection point, and positive thereafter. We next must ask what portion of the F - z curve is included in the domain of validity.

Suppose a line is drawn across Figure 4 at .63 units above the horizontal axis. Then either this line will intersect the hyperbola $\left[(8 - 2\frac{a}{b})z - 11\right] / \left[(4 + 2\frac{a}{b})a - 10\right]$ or else it will lie completely below the hyperbola. The latter case, which we consider first, corresponds to

$$\frac{1 - 4\frac{b}{a}}{1 + 2\frac{b}{a}} > .63$$

(5.11)

$$\frac{b}{a} < .070 \qquad .$$

In this case the cirve representing $c_{10}\, z^{1/3}\, e^{4/3\, z(\frac{a}{b} - 1)}$ will intersect the horizontal line before it intersects the hyperbola. Since the range of amplification is from $0 < c_{10}\, z^{1/3}\, e^{4/3\, z(\frac{a}{b} - 1)} < .63$, in this case the slope of the F - z curve is negative throughout the amplification region, since the minimum is outside.

Next let us suppose that the horizontal line intersects the hyperbola. The z - coordinate of the point of intersection is given by the root, z_1, of the equation:

$$.63 = - \frac{(8 - 2\frac{a}{b})z_1 - 11}{(4 + 2\frac{a}{b})z_1 - 10}$$

(5.12)

$$z_1 = \frac{17.3\frac{b}{a}}{10.5\frac{b}{a} - .74} \qquad .$$

When $\frac{b}{a} = .070$, z, is infinite, and it decreases to $z_1 = 3$ at $\frac{b}{a} = .156$. Let us analyze the behavior for $.070 < b/a < .156$. The curve

$c_{10}\, z^{1/3}\, e^{4/3\, z(\frac{a}{b} - 1)}$ will intersect the hyperbola either to the right or to the left of z_1, depending on the value of c_{10}. For a critical value of c_{10}, say c_{101}, the two curves and the horizontal line will meet in a single point. This critical value is given by;

$$c_{101} = .63\ z_1^{-1/3}\ e^{4/3\ z_1\ (\frac{a}{b} - 1)} \tag{5.13}$$

Now if $c_{10} < c_{101}$, the two curves intersect to the right of $z = z_1$. Now the domain of validity is from $z = 3$ to $z = \infty$, while the amplifying range is from $z = 0$ to the intersection of $c_{10}\ z^{1/3}\ e^{4/3\ z(\frac{a}{b} - 1)}$ and the horizontal line. Since the hyperbola is a monotone decreasing curve, its intersection with $c_{10}\ z^{1/3}\ e^{4/3\ z(\frac{a}{b} - 1)}$ has an ordinate less than .63, and the minimum is therefore within the amplifying range. Consequently, under these conditions the F - z curve will possess a minimum. However, if $c_{10} > c_{101}$, the two curves intersect to the left of $z = z_1$, and therefore at an ordinate greater than .63. Hence, the minimum point is outside the amplifying range, and the included portion of the F - z curve, will have a negative slope.

The next range to be considered is $.156 < b/a\ .25$. In this range z_1 is less than 3. Consequently, the only portion of the hyperbola to be considered, that for which $z > 3$, lies entirely in the amplifying range. Now there will be acritical value of c_{10}, say c_{102}, for which the two curves intersect at $z = 3$. This critical value is given by:

$$c_{102} = \frac{6\frac{a}{b} - 13}{6\frac{a}{b} + 2}\ 3^{-1/3}\ e^{-4(\frac{a}{b} - 1)} \tag{5.14}$$

If $c_{10} > c_{102}$, the curves intersect for a value of z less than 3. Accordingly, the minimum is outside the domain of validity, and the included portion of the F - z curve has a positive slope, since the minimum occurs to the left of the domain. If $c_{10} < c_{102}$, the curves intersect when z is greater than 3, so the F - z curve has a minimum within the domain of validity.

This brings us up to $b/a = 1/4$. For larger values of b/a, the hyperbola intersects the axis at the value $z = 11/(8 - 2\frac{a}{b})$, and then goes to the asymptotic value $-(4\frac{b}{a} - 1)/(2\frac{b}{a} + 1)$. Consequently, the intersection of the two curves must take place for z less than $11/(8 - 2\frac{a}{b})$. We therefore ask when the last expression equals 3, which occurs for $b/a = 6/13 = .462$. When $1/4 < b/a < 6/13$, the behavior of the F - z curve is the same as when $.156 < b/a < 1/4$, namely, characterized by a parameter c_{102}, such that the F - z curve has a minimum for $c_{10} < c_{102}$ and the F - z curve has positive slope for $c_{10} > c_{102}$.

Finally, suppose b/a is greater than 6/13. The intersection of the two curves will then occur when z is less than 3, so the minimum is to the left of the domain of validity and the F - z curve has a positive slope.

Sketches of the variation of noise figure with voltage for various values of b/a are given in Figure 5.

We now ask which of these conditions is likely to obtain in practice. We shall use some numbers cited by L. M. Field of Stanford University on the performance of a 20,000 MC traveling wave tube. This tube has a helix with .090" outer diameter, .013" wire diameter, and a length of 1.6 inches. The wavelength is 1.5 cm and synchronous velocity occurs when the voltage is 2700 volts. From the figures, we compute $x_o = 3.78$, so we are within the domain of validity only when b/a is quite close to unity, specifically, when $.8 < b/a < 1$. However, we have selected the domain of validity arbitrarily, by requiring the asymptotic expansions of the Bessel functions to represent the functions with great accuracy. By sacrificing some few percent of accuracy, we can bring the domain of validity down to z = 2 with not too great trepidation. This brings the allowed values of b/a down to .53. However, the entire range still lies in the range $b/a < .462$, so we can expect the last curve in Figure 6 to be valid and the noise figure decreases with beam voltage.

For this choice of parameters, $c_{10} = .0525\ I^{1/3}/(b/a)$. When b/a = .8, $c_{10k} = .161$, and for all reasonable currents c_{10} is much less than c_{10k}, so the amplifying region is broad. When b/a = .6, $c_{10} = c_{10k}$ for a current I = 41 ma, which is still too large for practice. Consequently, we need not worry much about the restriction $c_{10} < c_{10k}$, since it is generally satisfied in practice.

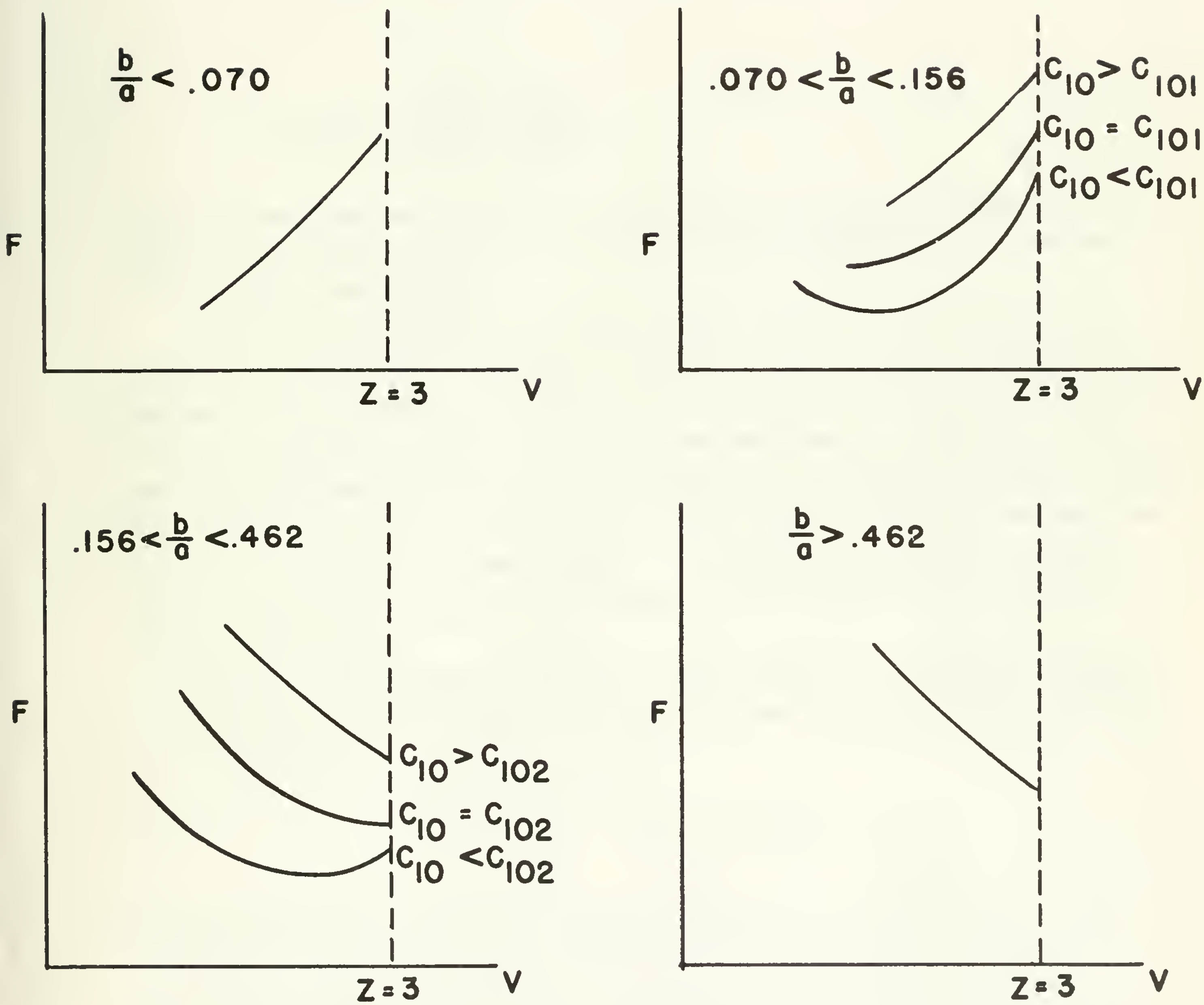
6. Dependence of Gain and Noise Figure on Beam Radius for Constant Current.

We shall now keep a, ω, I, and V fixed, and allow b to vary. Since a is fixed, we may take b/a as the independent variable. The work of this section corresponds to varying the beam radius while fixing the total current, so we effectively vary the current density. In the next section, we shall keep the current density $I/\pi b^2$ fixed, and vary the radius and current simultaneously. The latter conditions correspond to a variable cathode area and fixed focusing, while the present section refers to a fixed cathode area and variable focusing.

Equation (2.13) for the gain, when expressed as a function of b/a becomes:

Figure 5

Sketch of Variation of Noise Figure with Beam Voltage



$$G = c_{11} \frac{e^{2/3\, x_0 \frac{b}{a}}}{\left(\frac{b}{a}\right)^{2/3}} \left[1 - \frac{c_{12}\, e^{-4/3\, x_0 \frac{b}{a}}}{\left(\frac{b}{a}\right)^{2/3}}\right]^{1/2}$$

(6.1) $$c_{11} = \frac{\sqrt{3}}{4}\, c_1 \frac{I^{1/3}}{V^{1/2}} \frac{x_0^{1/3}\, e^{-2/3\, x_0}}{a}$$

$$c_{12} = \frac{2}{3}\, c_1 \frac{I^{1/3}}{V^{1/2}} \frac{e^{4/3\, x_0}}{x_0^{2/3}} \quad .$$

This function vanishes at the value of b/a, call it $(b/a)_m$, at which the radical vanishes, and rises from zero there with infinite slope. It is positive for b/a greater than $(b/a)_m$.

The domain of validity if from $b/a = 3/x_0$ to $b/a = 1$. The factor in G outside the radical has a minimum at $b/a = 1/x_0$, and increases monotonically thereafter. The radical also is monotone increasing, so the gain is a steadily increasing function of b/a. This is what one would expect physically, since the fields produced by the helix are strongest near the helical surface. Larger values of b/a mean that electrons are traveling nearer the helical surface, and hence in the region of stronger fields. We would therefore expect a greater transfer of energy between the fields and the beam, which manifests itself as greater amplification.

There are two possible conditions. Either $(b/a)_m$ lies within the domain of validity or it is exterior. The critical value, for which $(b/a)_m$ corresponds to $x_0 = 3$, occurs when c_{12} has the value

(6.2) $$c_{121} = \left(\frac{3}{x_0}\right)^{2/3} e^4 = 112\, x_0^{-2/3} \quad .$$

When c_{12} is less than c_{121}, $(b/a)_m$ is exterior to the domain of validity, so the gain is a monotone increasing function of b/a throughout. When c_{12} is greater than c_{121}, $(b/a)_m$ is greater than $3/x_0$, so the gain is zero for $3/x_0 < b/a < (b/a)_m$, and increases monotonically thereafter. Since b/a must be less than 1, the upper limit on c_{12} is given by:

(6.3) $$c_{12k} = e^{4/3\, x_0} \quad .$$

For still larger values of c_{12}, the gain vanishes throughout the domain of validity.

Let us now consider a practical case. For reasonable values of x_o and V, the factor multiplying $I^{1/3}$ in c_{12} will be on the order of one tenth. For $x_o = 5$, $V = 1400$ V, c_{12} becomes $.147\ I^{1/3}$, while c_{121} is 37. Consequently, c_{12} is much less than c_{121} for practical currents, and the gain increases monotonically with b/a throughout.

Similar considerations apply to the noise figure. Equation (3.5) for the noise figure becomes

$$F = c_{13}\ \frac{e^{8/3\ x_o \frac{b}{a}}}{\left(\frac{b}{a}\right)^{8/3}} \left[1 + c_{14}\ \frac{e^{-4/3\ x_o \frac{b}{a}}}{\left(\frac{b}{a}\right)^{2/3}}\right]$$

(6.4)

$$c_{13} = \frac{c_2\ VI^{1/3}\ e^{-2/3\ x_o}}{\omega\ a\ x_o^{5/3}}$$

$$c_{14} = \frac{1}{3}\ \frac{c_1\ I^{1/3}\ e^{4/3\ x_o}}{V^{1/2}\ x_o^{2/3}} \quad .$$

This function is the sum of the two functions $c_{13}\ e^{8/3\ x_o \frac{b}{a}}/(b/a)^{8/3}$ and $c_{13}\ c_{14}\ e^{4/3\ x_o \frac{b}{a}}/(b/a)^{10/3}$, each of which is monotone increasing when b/a is greater than $5/2x_o$. Since the domain of validity is from $b/a = 3/x_o$ to $b/a = 1$, the minimum occurs for a value of b/a outside the domain, and the noise figure increases steadily with b/a. We wish to limit ourselves to the amplifying range of the tube, and therefore should only draw the F - (b/a) curve from $(b/a)_m$ to $b/a = 1$, defining $(b/a)_m$ as that value of b/a for which the gain vanishes. However, in practical cases $(b/a)_m$ is well below the domain of validity, and we need not consider this modification.

7. Dependence of Gain and Noise Figure on Beam Radius at Constant Current Density.

We shall here keep a, ω, V, and I/b^2 constant, and regard b/a as the variable. Variations of b when I/b^2 is kept constant imply simultaneous variations in I. This section accordingly describes variations from tube to tube when the focusing conditions are kept fixed and the cathode is changed.

Equation (2.13) for the gain becomes:

$$G = c_{15}\, e^{2/3\, x_o \frac{b}{a}} \left[1 - c_{16}\, e^{-4/3\, x_o \frac{b}{a}}\right]^{1/2}$$

(7.1) $$c_{15} = \frac{\sqrt{3}}{4} \frac{c_1}{V^{1/2}} \left(\frac{I}{b^2}\right)^{1/3} \left(\frac{x_o}{a}\right)^{1/3} e^{-2/3\, x_o}$$

$$c_{16} = \frac{2}{3} \frac{c_1}{V^{1/2}} \left(\frac{I}{b^2}\right)^{1/3} \left(\frac{x_o}{a}\right)^{-2/3} e^{4/3\, x_o} \quad .$$

This function vanishes fo $b/a = (b/a)_m = \frac{3}{4x_o} \ln c_{16}$, and increases monotonically with b/a thereafter. This value $(b/a)_m$ will lie outside the domain of validity provided c_{16} is less than e^4, and consequently G increases steadily with b/a throughout the domain of validity. When c_{16} lies between e^4 and $e^{4/3\, x_o}$, $(b/a)_m$ lies in the domain, so the gain is zero up to $(b/a)_m$ and increases thereafter. When c_{16} is greater than $e^{4/3\, x_o}$, the gain is zero for all values of b/a.

Using the same numbers as before, c_{16} is small for all practical currents. Thus the gain will increase steadily with b/a. The physical reasons are the same as previously discussed in Section 6.

The noise figure follows the same pattern as in Section 6. Equation (3.5) becomes:

$$F = c_{17} \frac{e^{8/3\, x_o \frac{b}{a}}}{(b/a)^2} \left[1 + c_{18}\, e^{-4/3\, x_o \frac{b}{a}}\right]$$

(7.2) $$c_{17} = \frac{c_2 V}{\omega a^{1/3}} \left(\frac{I}{b^2}\right)^{1/3} \frac{e^{-2/3\, x_o}}{x_o^{5/3}}$$

$$c_{18} = \frac{1}{3} \frac{c_1}{V^{1/2}} \left(\frac{I}{b^2}\right)^{1/3} \left(\frac{x_o}{a}\right)^{-2/3} e^{4/3\, k_o} \quad .$$

Exactly as in Section 6, this function is the sum of two functions, each of which increases monotonically throughout the domain of validity. As before, we should only draw that portion of the curve which lies in the amplifying region. The critical value of c_{18} for which the gain vanishes at the edge of the domain of validity is $c_{18} = .63\, e^4$. For all practical tubes c_{18} is much less than this critical value, so the amplifying region is wider than the domain of validity.

8. Dependence of the Gain and Noise Figure on Frequency.

In this section, we shall keep a, b, V, and I fixed, and allow ω to vary. The gain only involves the frequency in the combination $z = bx_0/a = \omega b/v_0$, and we may use this expression to describe the frequency dependence. Since the resonance condition $d_o = 0$ may be written in the form

$$0 = d_o = \omega a\left[\frac{1}{v_o} - \frac{\cot\theta}{c}\right] \tag{8.1}$$

we see that by choosing the beam velocity $v_o = c \tan\theta$ we may satisfy the resonance condition for all ω. The situation is accordingly somewhat simpler than in the section involving the dependence on beam voltage, where we used z as variable and appealed to the broadness of the maximum of the curve representing u_1 as a function of d_o to neglect the departures from resonance.

Equation (2.13) for the gain becomes:

$$G = c_{19}\, z^{1/3}\, e^{-2/3\, z\, (\frac{a}{b} - 1)} \left[1 - c_{20}\ \frac{e^{4/3\, z(\frac{a}{b} - 1)}}{z^{2/3}}\right]^{1/2}$$

$$c_{19} = \frac{\sqrt{3}}{4}\ \frac{c_1}{b}\ \frac{I^{1/3}}{V^{1/2}} \tag{8.2}$$

$$c_{20} = \frac{2}{3}\ c_1\ \frac{I^{1/3}}{V^{1/2}} \quad .$$

This function vanishes at the two roots of the radical which we call z_{min} and z_{max}, and is positive between these two roots. The radical has a maximum for $z = 1/2\,(\frac{a}{b} - 1)$, and so does the multiplying factor, so the gain should display a maximum for this critical value of z. The gain at the maximum is

$$G_m = \frac{c_{19}}{\left[2e(\frac{a}{b} - 1)\right]^{1/3}} \left[1 - c_{20}\ (2e(\frac{a}{b} - 1)^{2/3}\right]^{1/2} . \tag{8.3}$$

The maximum occurs in the domain of validity if $1/2(\frac{a}{b} - 1)$ is greater than 3, which requires b/a greater than 6/7. When b/a is less than 6/7, the maximum lies outside the domain, so the gain will be a monotone decreasing function of z throughout the admissible range, vanishing at z_{max}. The con-

dition that z_{max} also lie in the domain requires c_{20} to be less than $3^{2/3}\, e^{-4\,(\frac{a}{b} - 1)}$. If c_{20} is greater than this critical value, there will be no amplification in the domain.

If b/a is greater than 6/7, the G - z curve has a maximum in the domain, provided again that z_{max} is greater than 3. When c_{20} is less than $3^{2/3}\, e^{-4(\frac{a}{b} - 1)}$, G has a positive value at z = 3, increases to the maximum value G_m, and then decreases to zero at z_{max}. When c_{20} lies between $3^{2/3}\, e^{-4(\frac{a}{b} - 1)}$ and $\left[2e(\frac{a}{b} - 1)\right]^{-2/3}$, z_{min} also lies in the domain, so G is zero up to z_{min}, then increases to the maximum and falls to zero at z_{max}. When c_{20} is greater than $\left[2e(\frac{a}{b} - 1)\right]^{-2/3}$, equation (8.3) yields as imaginary value for the maximum gain, so the tube does not amplify at all. This is the same limiting current behavior that was discussed earlier.

The bandwidth of the tube is determined by c_{20}. We define the bandwidth as the separation between the values of frequency where the power gain per unit length is half its maximum value. Let us introduce a variable μ, defined by $\mu = z^{-1/3}\, e^{2/3\, z(\frac{a}{b} - 1)}$. Then the half-power points are determined by $\mu_{1/2} = 2\mu_o \left[1 + 3\, c_{20}\, \mu_o^2\right]^{-1/2}$, where $\mu_o = \left[2e(\frac{a}{b} - 1)\right]^{1/3}$ is the value of μ at maximum gain, and we wish to find the associated values of z. These will be the roots of the equation:

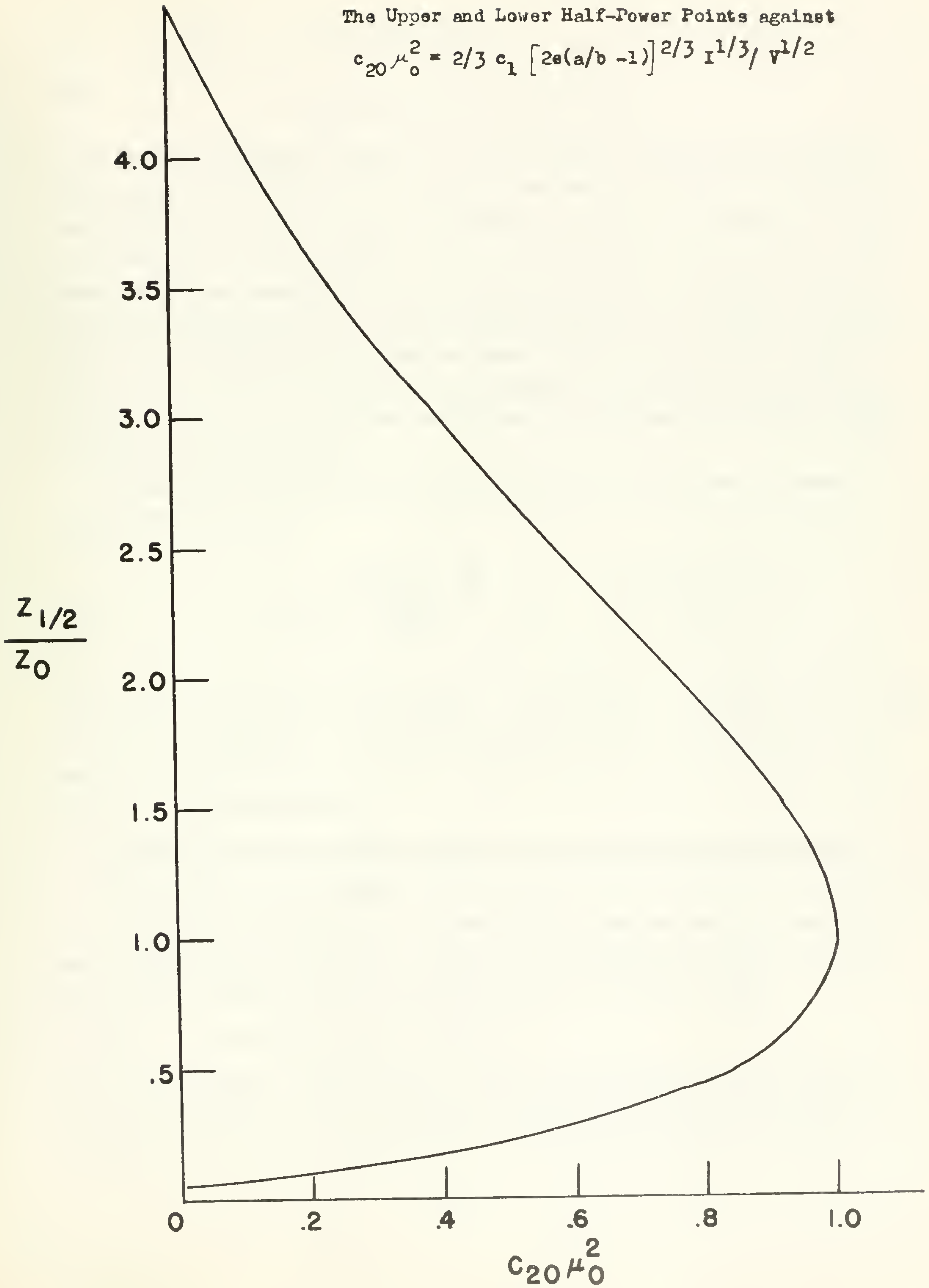
$$\frac{e^{z/z_o}}{z/z_o} = e \left(\frac{\mu_{1/2}}{\mu_o}\right)^3 \tag{8.4}$$

where $z_o = 1/2\ (\frac{a}{b} - 1)$. Since the amplification range is limited to $c_{20}\mu_o^2 < 1$, the range of $\mu_{1/2}/\mu_o$ is from 2 at $c_{20}\mu_o^2 = 0$ to 1 at $c_{20}\mu_o^2 = 1$. We have plotted the roots of (8.4) against $c_{20}\,\mu_o^2$ in Figure 6. Since c_{20} is proportional to $I^{1/3}/V^{1/2}$, Figure 6 gives the bandwidth as a function of voltage and current. It is clear that the relative bandwidth, which is the difference between the ordinates of the two curves in Figure 6, is a monotone decreasing function of $c_{20}\mu_o^2$, and thus of $I^{1/3}/V^{1/2}$. The maximum bandwidth, for $c_{20}\mu_o^2 = 0$, is 4.55, and an approximate formula valid when $c_{20}\mu_o^2$ is small is:

Figure 6

The Upper and Lower Half-Power Points against

$c_{20}\mu_o^2 = 2/3\, c_1 \left[2e(a/b - 1)\right]^{2/3} I^{1/3} / V^{1/2}$



$$\frac{\Delta\omega}{\omega_0} = 4.55 - 5.53\, c_{20}\mu_o^2$$

(8.5)
$$= 4.55 - 165\, I^{1/3} V^{-1/2} \left(\frac{a}{b} - 1\right)^{2/3} .$$

It is clear from Figure 6 that most of the bandwidth occurs at the high-frequency side. The product of the gain and the bandwidth, which is a measure of the effectiveness of the tube as an amplifier, is plotted in Figure 7. Since the bandwidth is monotone decreasing, the product GB looks very similar to G. Since the maximum gain occurs at $c_{20}\mu_o^2 = 2/3$, Figure 7 shows that when one operates at maximum gain one is sacrificing bandwidth, and a better operating point would be $c_{20}\mu_o^2 = .45$, where the gain-bandwidth product is maximum. However, oscillation will usually set in before the current corresponding to this value can be reached unless ocillation is eliminated by providing lumped or distributed attenuation.

We next shall consider the frequency dependence of the noise figure. Equation (3.5) yields:

(8.6)
$$F = c_{21} \frac{e^{2/3\, z(4 - \frac{a}{b})}}{z^{8/3}} \left[1 + \frac{c_{22} e^{4/3\, z(\frac{a}{b} - 1)}}{z^{2/3}} \right] .$$

The analysis of this expression is virtually identical with that of Section 5, except for some changes in the numbers. F decreases to a minimum of some value of z and then increases. The analysis discusses when the minimum lies in the domain of validity and when it lies in the amplification range. There seems little point in presenting it.

9. Effect on the Gain and Noise Figure of Small Departures from Resonance.

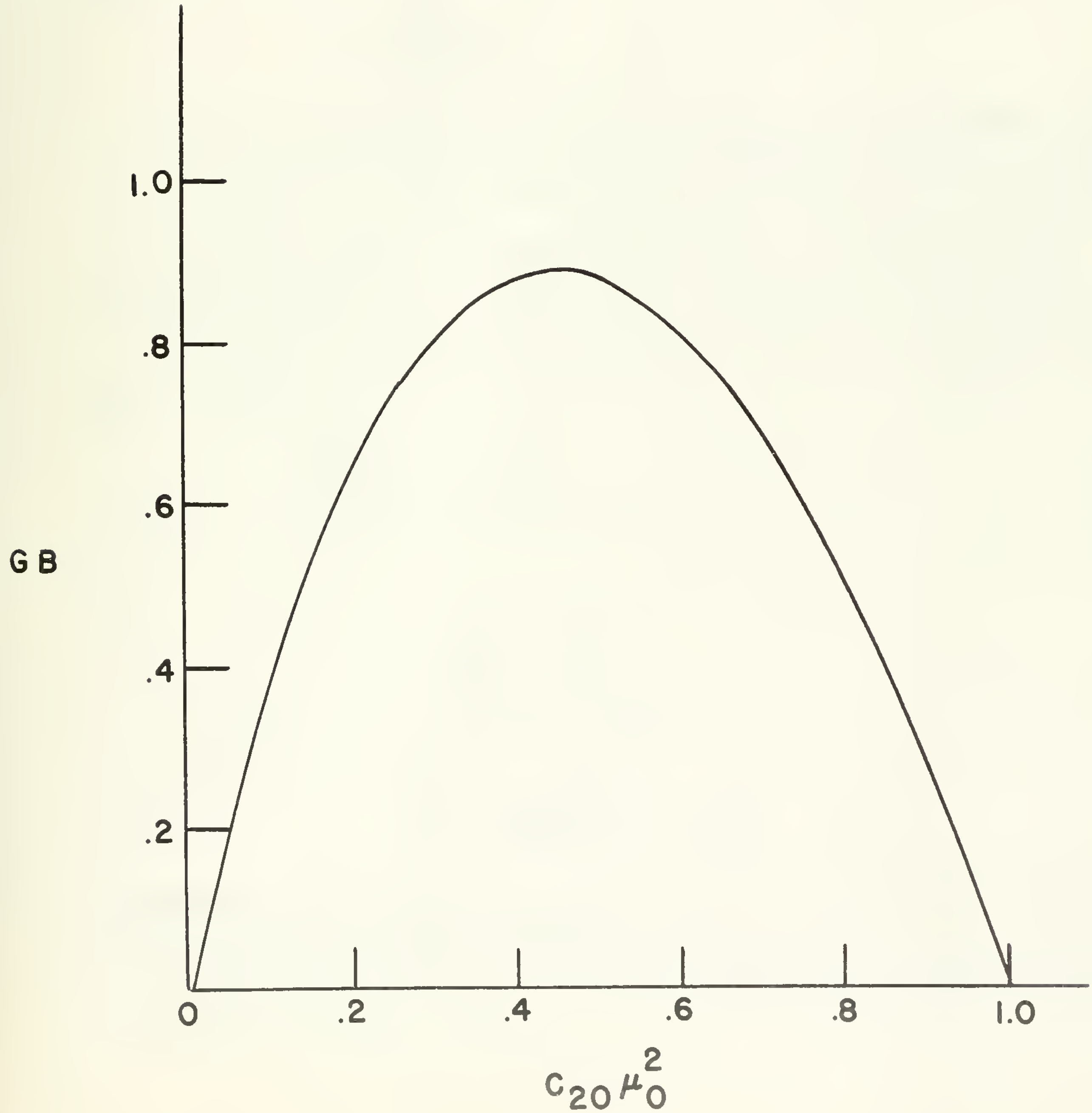
Heretofore we have assumed for the most part that the parameters satisfied the resonance condition $d_o = 0$. We now wish to consider the case that d_o is small but different from zero. To obtain the propagation constants in this case let us consider the first cubic equation of Friedman, namely,

(2.5)
$$y(y - d_o)^2 - \zeta(1 + \int y) = 0 .$$

If d_o is permitted to vary while ζ and $\int$ are maintained constant, then y will be a function of d_o. We will regard y as a perturbation on y_o, the value of y for $d_o = 0$, and will find the first order correction term. Let $y = y_o + e\, d_o$

Figure 7

The Normalized Gain-Bandwidth Product against $c_{20}\mu_o^2$



where s is to be determined. Substituting this in Eq. (2.5) and neglecting all powers of d_o higher than the first, we find

$$(9.1) \qquad s = \frac{2\,y_o^2}{3y_o^2 - \rho\xi} \quad .$$

Actually the value of s depends upon which of the three roots of the cubic we are considering. To emphasize this we rewrite Eq. (9.1) as follows

$$(9.2) \qquad \begin{aligned} y_i &= y_{oi} + s_i\, d_o \\ s_i &= \frac{2y_{oi}^2}{3y_{oi}^2 - \rho\xi} \end{aligned} \qquad i = 1,\ 2,\ \text{or } 3 \quad .$$

From the last equation it follows, since $\mathrm{Im}\, y_{o1} > 0$ by definition, and $\rho\xi = L > 0$ from Eq. (2.8), that $\mathrm{Im}\, s_1 < 0$. Thus $\mathrm{Im}\, y_1$, and hence the amplification increases as V increases (d_o decreases), while a, b, ω, θ, and $I/V^{3/2}$ are maintained constant (insuring constancy of ρ, ξ, and therefore of y_o). The maximum amplification will occur for some negative value of d_o implying the tube should be operated slightly below resonance. We next consider the noise figure.

If we use Eq. (9.2) in Eq. (3.3), we obtain for the noise figure to the first order in d_o

$$(9.3) \qquad F = \frac{4\Gamma^2 m^2 v_o^6}{\gamma k_1 e T \omega^2 I}\ \frac{k e^{2x_o}}{a x_o^3}\ \left| y_{o2}\, y_{o3} \right|^2 \left(1 + 2\, d_o\, E\right)$$

where

$$(9.4) \qquad E \equiv \mathrm{Re}\left[\frac{s_2 - 1}{y_{o2}} + \frac{s_3 - 1}{y_{o3}}\right] \quad .$$

From Eqs. (2.10) and (9.2) it follows that

$$(9.5) \qquad E = \frac{(A/L)^{1/3}}{6}\ (1 - 4B) \quad .$$

Consequently, E may take on negative or arbitrarily large and positive values if the parameters are suitably chosen.

If now we keep a, b, ω, θ, and $I/V^{3/2}$ constant in Eq. (9.3) and permit v_o to vary, we obtain

(9.6) $$F = c_{23} v_o^3 (1 + 2 d_o E) \quad ,$$

whose logarithmic derivative is

(9.7) $$\frac{1}{F} \frac{dF}{dv_o} = \frac{3}{v_o} - \frac{2E\omega a}{v_o^2(1 + 2 d_o E)} \quad .$$

At $d_o = 0$, this has the value

(9.8) $$\frac{1}{F} \frac{dF}{dv_o}\bigg|_{d_o = 0} = \frac{3}{v_{oo}} - \frac{2E\omega a}{v_{oo}^2}$$

where v_{oo} is given by $v_{oo} = \omega a/x_o$. Since F is always positive and E may be negative or arbitrarily large and positive, it follows from Eq. (9.8) that F may either increase or decrease as v_o is increased above v_{oo}, depending on the values selected for the fixed parameters.

10. Extension of the Domain of Validity.

Throughout Sections 4 - 9 we have assumed that all the Bessel functions originally involved in the equations have been replaced by their asymptotic forms. We shall now consider the effect of lifting this restriction. We therefore return to Equation (2.5) for the gain, which we rewrite with some slight changes of notation:

(10.1)
$$\begin{aligned}
&y(y - d_o)^2 - \mathcal{f}(x_o, z)\, \mathcal{g}(x_o, z)\, W y + \mathcal{g}(x_o, z)\, W = 0 \\
&\mathcal{f}(x_o, z) = -\frac{K_o(z)}{I_o(z)}\, G'(x_o) \\
&\mathcal{g}(x_o, z) = -\frac{x_o^2}{2G'(x_o)} \left[I_o^2(z) - I_1^2(z) \right] \\
&W = I/2\pi\epsilon\sqrt{2e/m}\ \ V^{3/2} = c_1^3\, I/V^{3/2} \\
&z = bx_o/a
\end{aligned}$$

The functions $\mathcal{f}(x_o, z)$ and $\mathcal{g}(x_o, z)$ as defined here are positive, since G' is negative. Since $\mathcal{f}$ and $\mathcal{g}$ are each factored into a product of a function of x_o and a function of z, it is possible to construct universal curves for the separated factors and use these for any choice of the parameters.

We set d_0 equal to zero, thereby considering only the resonance behavior, and then we observe that the equation (10.1) may be thrown into the form (2.9) by the transformation

$$y = \left[\zeta(x_0, z)\, W \right]^{1/3} u \tag{10.2}$$

whence we obtain the standard form (2.9), repeated here:

$$u^3 - B\,u + 1 = 0 \tag{2.9}$$

where the parameter B is given by:

$$B = \xi(x_0, z) \left[\zeta(x_0, z)\, W \right]^{1/3}$$

$$= \frac{c_1}{\sqrt[3]{2}} \left[-x_0\, G'(x_0) \right]^{2/3} \frac{K_0(z)}{I_0(z)} \left[I_0^2(z) - I_1^2(z) \right]^{1/3} \frac{I^{1/3}}{V^{1/2}}. \tag{10.3}$$

The previous analysis was based upon the assumption that the arguments of all the Bessel functions are large, in which case B has the asymptotic form:

$$B_\infty = c_1\, e^{4/3\, x_0} \frac{e^{-4/3\, z}}{z^{2/3}} \frac{I^{1/3}}{V^{1/2}} \quad . \tag{10.4}$$

We now ask how much B departs from its asymptotic value as x_0 and z decrease. Detailed numerical analysis shows that $-x_0\, G\,(x_0)$ differs from its asymptotic form $\frac{2}{\pi} e^{2x_0}$ by only 15% over the range x_0 greater than .1, so all conclusions derived from considering that particular factor of B may be extended to the domain $x_0 > .1$ with only a small error.

The z-dependent factor of B, which we call B_z, is somewhat more complicated. In view of the overwhelming effect of the exponential factor $e^{-4/3\, z}$, we have calculated the departure of $B_z\, e^{4/3\, z}$ from its asymptotic form $\pi^{2/3}/2^{1/3}\, z^{2/3} = 1.705/z^{2/3}$ over the range $.1 < z < 5$. Curves of the two functions are given in Figure 8. The error is 8% at $z = 2$, 22% at $z = 1$, and 45% at $z = .5$. Consequently, use of the asymptotic form will be markedly in error when z is less than 2. In this case, the exact curve given in Figure 8 may be used. It is plain that the exact B_z increases much more slowly than the asymptotic form as z decreases. A good approximation over the range $.05 < z < .5$ ix $B_z = 1.7\, z^{-.3}$, which fits the curve quite accurately.

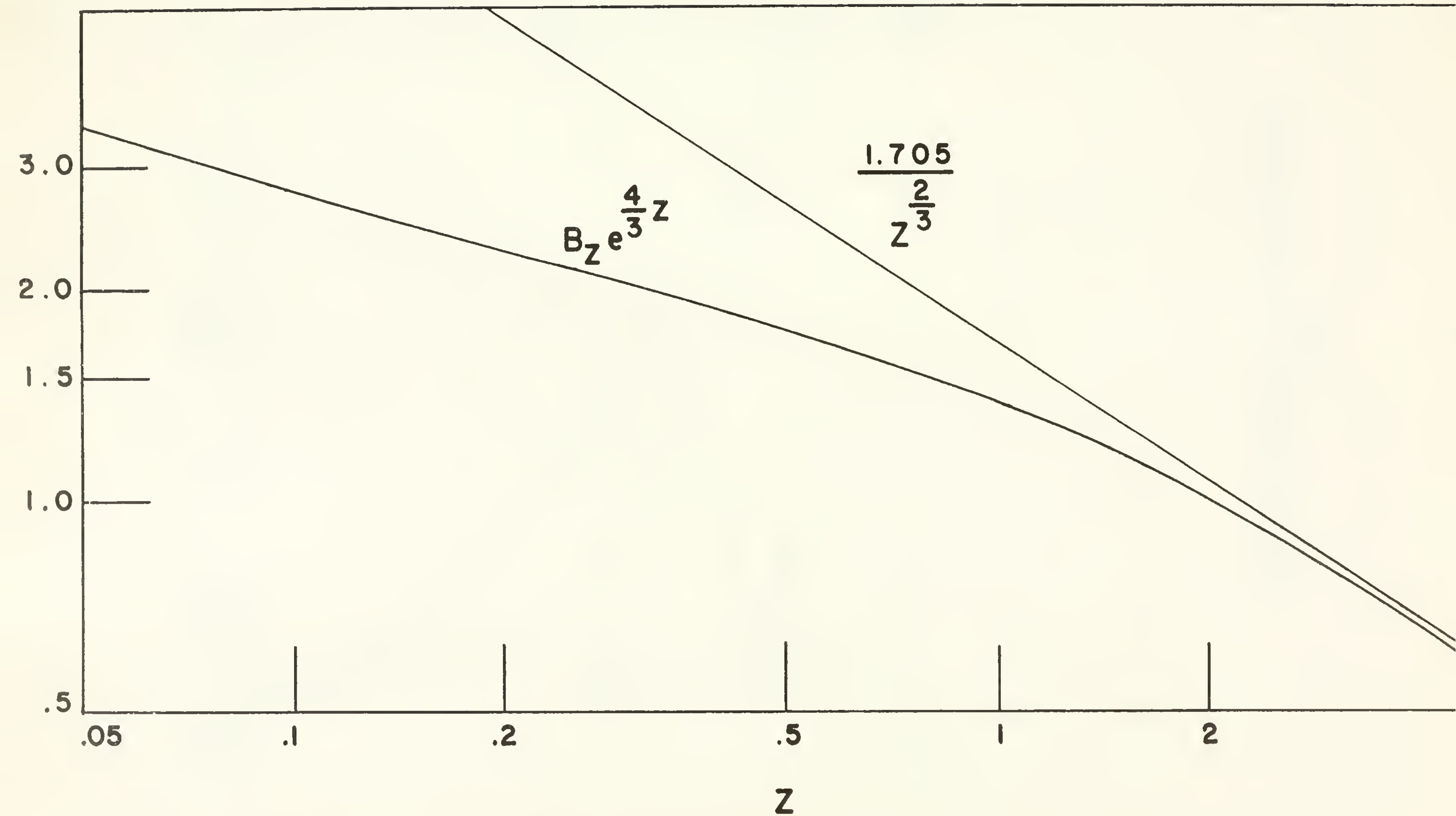


Figure 8

The z-dependent part of B versus z and its Asymptotic Form

Figure 9

The Exact Solution of the Cubic (2.9) versus the Parameter B over the Amplifying Range

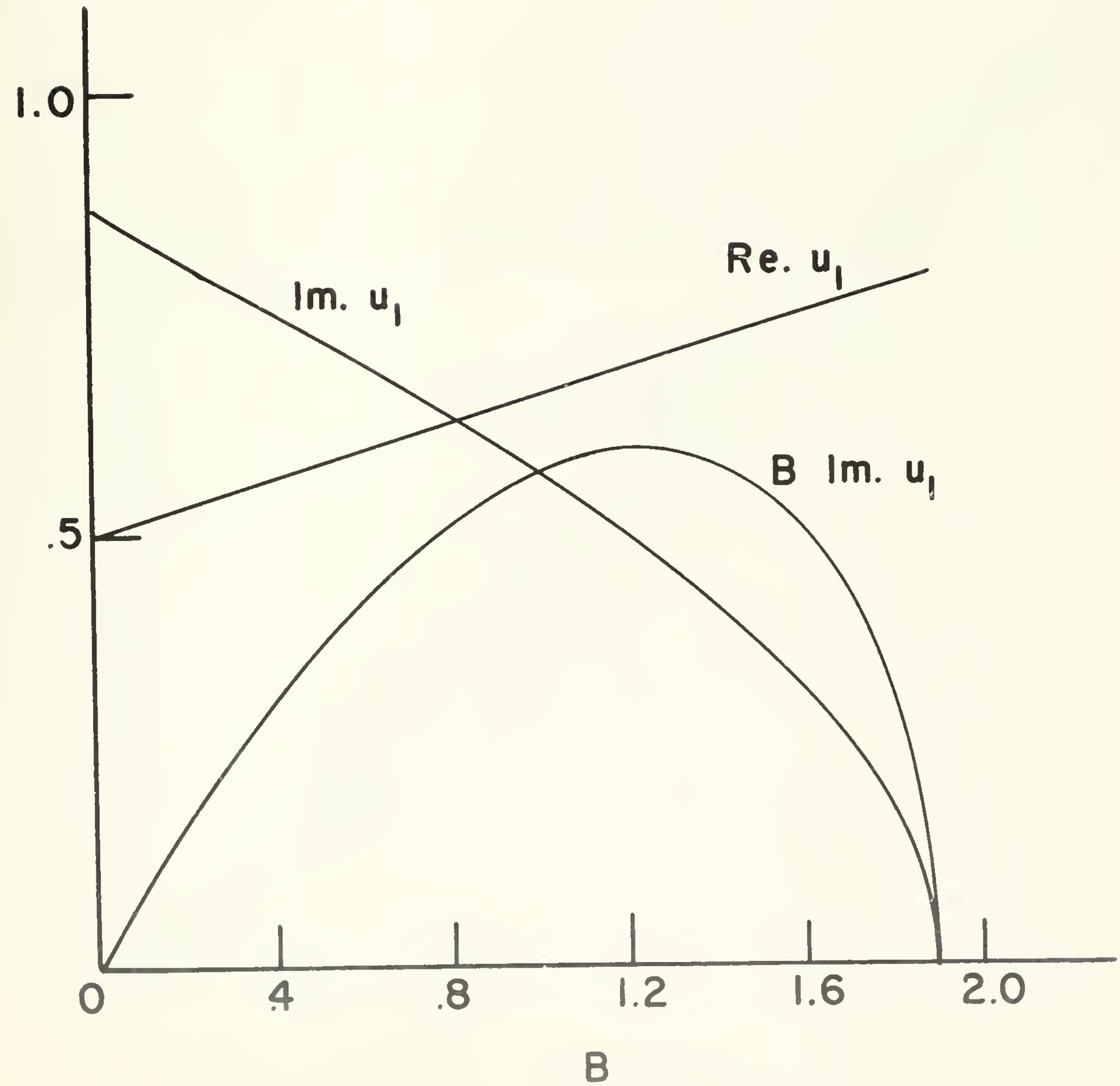
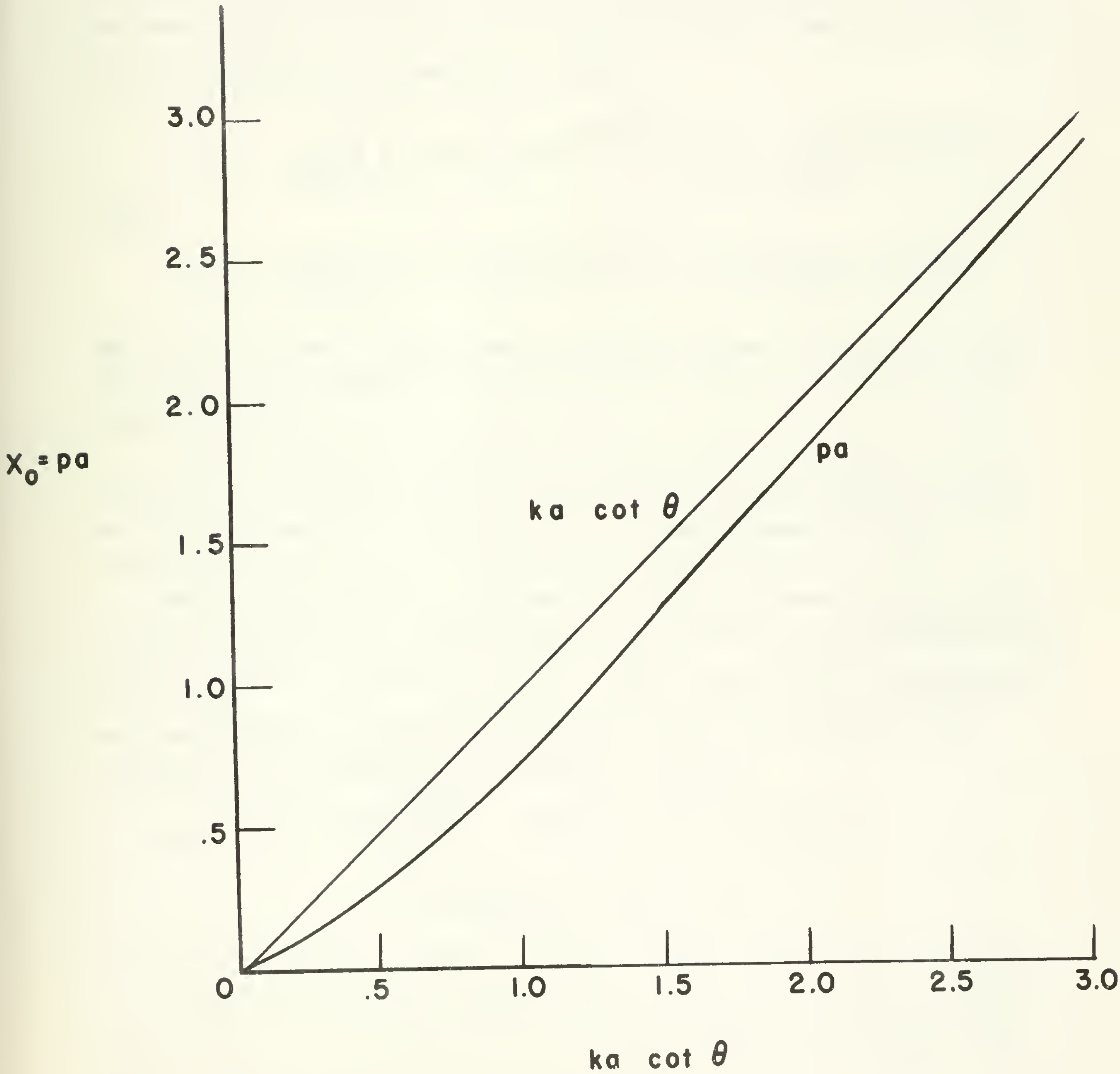


Figure 10

The Exact Solution of the Equation $G(x_o) = 0$ against ka cot θ



The quantity B can accordingly be determined over the extended range of parameters by the approximation, intermediate between (10.3) and (10.4)

$$B = \frac{c_1}{1.7}\, e^{4/3\, x_0}\, B_z(z)\, \frac{I^{1/3}}{V^{1/2}} = 18.4\, e^{4/3\, x_0}\, B_z\, \frac{I^{1/3}}{V^{1/2}} \tag{10.5}$$

where B_z is to be taken from the curve of Figure 8. Now having B, one may find the exact solution of the cubic (2.9) by use of Cardan's formula, which yields for the amplified mode in the case $B < 3/\sqrt[3]{4}$:

$$\begin{aligned} \mathrm{Re}\, u_1 &= \frac{1}{2^{4/3}} \left[\left\{ 1 + \left(1 - \frac{4}{27} B^3\right)^{1/2} \right\}^{1/3} + \left\{ 1 - \left(1 - \frac{4}{27} B^3\right)^{1/2} \right\}^{1/3} \right] \\ \mathrm{Im}\, u_1 &= \frac{\sqrt{3}}{2^{4/3}} \left[\left\{ 1 + \left(1 - \frac{4}{27} B^3\right)^{1/2} \right\}^{1/3} - \left\{ 1 - \left(1 - \frac{4}{27} B^3\right)^{1/2} \right\}^{1/3} \right] . \end{aligned} \tag{10.6}$$

The gain of the tube is proportional to $B\, \mathrm{Im}\, u_1$. Curves of $\mathrm{Re}\, u_1$, $\mathrm{Im}\, u_1$, and $B\, \mathrm{Im}\, u_1$ are given in Figure 9. The maximum of $B\, \mathrm{Im}\, u_1$ lies at $B = 1.24$, and has the value .58.

A further extension is obtained by using the exact value of the root x_0, instead of the approximate value $x_0 = ka \cot \theta$. Since the asymptotic values may be used above $x_0 = 3$, we have plotted the exact solution of $G(x_0) = 0$ against $ka \cot \theta$ for $ka \cot \theta$ less than 3 in Figure 10.

The following systematic procedure may be used to compute the gain. From the known values of ω, a, and θ, compute $ka \cot \theta$. Find $pa = x_0$ from Figure 10. Compute $\frac{2}{\pi x_0} e^{2x_0}$ and $e^{4/3\, x_0}$. Compute $z = \frac{b}{a} x_0$. Find $B_z\, e^{4/3\, z}$ from Figure 8, and then compute B_z. With these quantities, plus I and V, compute B from equation (10.5). Find $B\, \mathrm{Im}\, u_1$ from Figure 9. Compute

$\xi(x_0, z) = \frac{2}{\pi x_0}\, e^{2x_0}\, \frac{K_0(z)}{I_0(z)}$. The gain is finally given by;

$$G = \frac{1}{2a}\, \frac{1}{\xi(x_0, z)}\, B\, \mathrm{Im}\, u_1 \quad . \tag{10.7}$$

The noise figure may be treated in a similar manner. Since B is proportional to $I^{1/3}$, the curve of $B\, \mathrm{Im}\, u_1$ in Figure 9 gives directly the variation of the gain with current. We note finally that the linear increase of $\mathrm{Re}\, u_1$ with B corresponds to a steady detuning of the tube as the beam current is increased. This should serve to reduce the amplification somewhat.

Date Due
